Linear Algebra, Winter 2022

List 2

Lines and planes

29. Calculate the distance between the points $(5,1,1)$ and $(-8,0,7)$.

The **cross product** of $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$ is written $\vec{a} \times \vec{b}$. It is the vector that is perpendicular to $\vec{a}$ and $\vec{b}$, has length $|\vec{a}||\vec{b}|\sin(\theta)$, where θ is the angle between $\vec{a}$ and $\vec{b}$, and whose direction obeys the Right-Hand Rule.

To calculate $\vec{a} \times \vec{b}$, we can use $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, $\hat{k} \times \hat{\imath} = \hat{\jmath}$ and very careful algebra, or use the direct formula

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\hat{\imath} + (a_3b_1 - a_1b_3)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k}.$$

30. (a) Give an example of a vector parallel to $9\hat{\imath} + 6\hat{\jmath} + 2\hat{k}$ that has magnitude 2. How many vectors with those two properties exist?

    (b) Give an example of a vector perpendicular to $9\hat{\imath} + 6\hat{\jmath} + 2\hat{k}$ that has magnitude 2. How many vectors with those two properties exist?

    (c) Give an example of a vector that is perpendicular to $9\hat{\imath} + 6\hat{\jmath} + 2\hat{k}$ and also perpendicular to $5\hat{\imath} + 5\hat{k}$ and has magnitude 2. How many vectors with those three properties exist?

The vector $\vec{r}$ is used for $[x, y]$ in 2D and $[x, y, z]$ in 3D.

31. Re-write $\begin{cases} x = 2 + 5t \\ y = 3t \\ z = 9 \end{cases}$ as a single equation using vectors.

32. Describe in words the set of points (x, y) that satisfy

    (a) $y = x^2$.

    (b) $x^2 + y^2 = 100$.

    (c) $|x\hat{\imath} + y\hat{\jmath}| = 100$.

    (d) $|\vec{r}| = 100$.

    (e) $3x + 2y = 0$.

    (f) $\begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 0$.

    (g) $(3\hat{\imath} + 2\hat{\jmath}) \cdot \vec{r} = 0$.

A vector that is parallel to a line is called a **direction vector** for the line.

The line through (x_0, y_0, z_0) with direction $\vec{d} = [a, b, c]$ has "vector equation"

$$\vec{r} = \vec{p} + t\,\vec{d}.$$

To describe it without vectors we can use the *three* "parametric equations"

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

33. Describe the line through the point $(2,\ 2.4,\ 3.5)$ and parallel to the vector $3\,\hat{\imath} + 2\,\hat{\jmath} - \hat{k}$ using the parameter t in

    (a) one vector equation.

    (b) three scalar equations.

34. Write an equation for the line through points $(2, 3, 1)$ and $(-3, 7, 0)$.

35. Determine whether the point $(9, -10, 3)$ is on the line

$$x = 5 + 2t, \qquad y = 2 - 6t, \qquad z = -1 - t.$$

This is the same as asking whether there is a single value of t for which

$$9 = 5 + 2t \quad \text{and} \quad -10 = 2 - 6t \quad \text{and} \quad 3 = -1 - t.$$

36. Is the point $(4, 8, 7)$ on the line $\vec{r} = [1, 2, 6] + [3, 8, 9]t$?

37. Find the point where the lines

$$\begin{aligned} L_1 : &\quad x = 35 + 2t,\ y = 9 + t,\ z = -24 - 4t \\ L_2 : &\quad x = -3 + 2s,\ y = 16 - 3s,\ z = 13 + 2s \end{aligned}$$

intersect. That is, find one value of t and one value of s such that

$$35 + 2t = -3 + 2s \quad \text{and} \quad 9 + t = 16 - 3s \quad \text{and} \quad -24 - 4t = 13 + 2s$$

and then either use t to calculate the (x, y, z)-coordinates of the point from L_1 or use s to calculate the (x, y, z)-coordinates of the point from L_2.

38. Do the lines

$$\begin{aligned} L_1 : &\quad x = 3 + t,\ y = 2 - 2t,\ z = 1 + 5t \\ L_2 : &\quad x = 8 - 3s,\ y = 1 + s,\ z = -10 + 5s \end{aligned}$$

intersect? If so, find the intersection point and the angle between the two lines at that point (using a calculator for $\cos^{-1}$).

39. Do the lines

$$\begin{aligned} L_1 : &\quad x = 3 + t,\ y = 2 - 2t,\ z = 1 + 5t \\ L_3 : &\quad x = 5 + 2s,\ y = -6 - s,\ z = 7 - 4s \end{aligned}$$

intersect? If so, find the intersection point and the angle between the two lines at that point (using a calculator for $\cos^{-1}$).

40. Explain why the parametric equations

$$\begin{cases} x = 1 - t \\ y = 2 - 3t \\ z = 4t \end{cases} \quad \text{and} \quad \begin{cases} x = 2s \\ y = -1 + 6s \\ z = 4 - 8s \end{cases}$$

describe the same line.

A vector that is perpendicular to a plane is called a **normal vector** for the plane.

The plane through point (x_0, y_0, z_0) with normal vector $\vec{n} = [a, b, c]$ has

"vector equation" $\quad \vec{n} \cdot (\vec{r} - \vec{p}) = 0$

"scalar equation" $\quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

"standard equation" $\quad ax + by + cz = d$

where $\vec{p} = [x_0, y_0, z_0]$ and $d = \vec{n} \cdot \vec{p}$.

41. Find a scalar equation for the plane through the origin perpendicular to the vector $[1, -2, 5]$.

42. Find an equation for the plane through the point $(1, -1, -1)$ parallel to the plane $5x - y - z = 6$.

43. Find the intersection point of the line L and the plane P, where

$$\begin{aligned} L: &\quad x = t,\ y = 1 - 2t,\ z = -3 + 2t \\ P: &\quad 3x - y - 2z = 3. \end{aligned}$$

44. Find the distance between the point $(-6, 3, 5)$ and the plane $x - 2y - 4z = 8$ using the following steps:

    (a) Give a vector that is perpendicular to the plane $x - 2y - 4z = 8$.

    (b) Give an equation for the line through $(-6, 3, 5)$ perpendicular to the plane $x - 2y - 4z = 8$ (that is, parallel to the vector from part (a)).

    (c) Find the point where the plane $x - 2y - 4z = 8$ and the line from part (b) intersect.

    (d) Calculate the distance between $(-6, 3, 5)$ and the point from part (c). This is exactly the distance between $(-6, 3, 5)$ and the plane $x - 2y - 4z = 8$.

45. Consider the planes

$$\begin{aligned} P_1: &\quad 8(x-1) + 6(y+3) + 16(z+7) = 0, \\ P_2: &\quad 4x + 3y + 8z = 27. \end{aligned}$$

    (a) Do the planes intersect?

    (b) If the planes intersect, find the angle between the planes.

    (c) If the planes intersect, give an equation for the line that is their intersection.

46. Consider the planes

$$\begin{aligned} P_1: &\quad 8(x-1) + 6(y+3) + 16(z+7) = 0, \\ P_2: &\quad 2x - 3y + 7z = 1. \end{aligned}$$

    (a) Do the planes intersect?

    (b) If the planes intersect, find the angle between the planes.

    (c) If the planes intersect, give an equation for the line that is their intersection.

47. Find parametric equations for the line that is the intersection of the two planes $x + y = 3$ and $y + z = 1$.

48. Find the angle between the line and the plane (at the point where they intersect):

$$\begin{aligned} L: &\quad x = 5 + \sqrt{3}t,\ y = \sqrt{5} + 3t,\ z = -1 + 2t; \\ P: &\quad (x+6) + \sqrt{3}(y-4) + 2\sqrt{3}\,z = 0. \end{aligned}$$

49. If a is a scalar, $\vec{b}$ is a 2D vector, and $\vec{c}$ is a 3D vector, which of the following calculations are possible?

(a) $5 + a$

(b) $5 + \vec{b}$

(c) $5a$

(d) $5\vec{c}$

(e) $\vec{b} + \vec{c}$

(f) $\vec{b} \cdot \vec{c}$

(g) $\dfrac{\vec{c}}{5}$

(h) $\dfrac{\vec{c}}{|\vec{b}|}$

(i) $\dfrac{\vec{c}}{\vec{b}}$

(j) $\dfrac{\vec{c}}{\vec{c}}$

(k) $\vec{a}^3$

(l) $5 + |\vec{c}|^3$

(m) $|5\vec{a}|^3$

(n) $|\vec{a}| + |\vec{c}|$

(o) $|\vec{c}|\vec{a}$